Application of New Figures of Merit for Multi-Stage Cryocoolers

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ABSTRACT

Evaluation of the overall performance of a multi-stage cryocooler, including the cooling power available on the intermediate stages, is not trivial, but is necessary to choose or design efficiently a cooler for a defined application. We have recently proposed two simple definitions of electrical "figures of merit" (FoM) for multi-stage cryocoolers that represent the distributed refrigeration power for applications where heat-sinking of power and signal leads at intermediate stages is available. These FoMs are designed to have universal applicability and to convey the relative performance of such cryocoolers in a much more effective manner than simple weighted averages based on the coefficient of performance (CoP) of the various stages. We demonstrate the utility of these FoMs by applying them to a four-stage cryocooler over a broad range of temperatures, enabling us to determine the optimal operating temperature and heat lift of each stage of the cooler. We also show, using properties of a real superconducting electronic system operating at 4 K, how these FoMs can be used to help in the specification and design of an improved multi-stage cryocooler to fit efficiently the needs of such a system.

INTRODUCTION

Cryocoolers designed for temperatures near 4 K invariably require two, three, or even four cooling stages. Consider a system with n stages, each with steady-state operating temperatures T_i and excess heat lift Q_i. Generally, the device of interest is located at the coldest (n^{th}) stage. and it is conventional to evaluate the cryocooler performance in terms of this coldest stage, and the total room-temperature operating power P_{RT}, through the Coefficient of Performance CoP, which may be directly compared to the Carnot efficiency of the cooler.

$$CoP = Q_n / P_{RT} \tag{1}$$

Other approaches to evaluate the performances of a cryocooler and to optimize its parameters have been proposed [1,2,3]. However, these analyses are based on thermodynamical properties of the cooling system, and thus depend on a specific system and do not consider the end user application. Also, these analyses require a complete and thorough knowledge of the behaviour of the cooler and are thus not easily applied to a random cooler. For many systems where the thermal balance is dominated not by heat dissipation at the lowest temperature stage, but rather by heat conduction on electrical leads, it is better to introduce a more accurate yet as

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simple figure of merit than the CoP, for a multi-stage cryocooler including different cooling powers Q_i on the intermediate stages. In a previous article [4], we showed that two electrical figures of merit can be distinguished: A Power Lead FoM (PL-FoM) that gives the maximum current that can be carried down to the operating temperature, and a Signal Lead FoM (SL-FoM) that represents the minimum signal attenuation between room temperature and operating temperature. In both cases, it is assumed that the leads are composed of a normal metal that obeys the well known Wiedemann-Franz law (governing proportionality between electrical and thermal conductivity), with lead resistances optimized to minimize heat loads, and that all leads are well thermalized at each intermediate stage.

In many cryogenic systems, the largest contribution to heat load is due to heat carried on current bias or power leads, rather than that dissipated by the cryogenic devices themselves. The determination of the power-lead FoM is based on the fact that there is an optimization of the current leads running between two stages of a cooler, regardless of the number and composition of the leads [5], depending on the balance between Joule heating and thermal conduction. From these optimized leads and the cooling power available on the lowest temperature stage, it is possible to determine the maximum current I_i that can flow between two consecutive stages. The maximum current can be considered as a performance rating of the ith stage of the cooler. The overall rating I_{eff} of the multi-stage cryocooler is thus the minimum I_i of all stages, corresponding to the maximum current that can flow across all the stages of the cooler without exceeding the heat lift on any stage.

$$I_{eff} = \min(I_i) = \min\left\{ Q_i e\sqrt{3} \Big/ \left(2\pi\, k_B \sqrt{T_{i-1}^2 - T_i^2}\right) \right\} \approx \min\left\{ Q_i e \Big/ \left(3.6\, k_B T_{i-1}\right) \right\} \qquad (2)$$

where for i=1, $T_{i-1} = T_0$ is room temperature, and the approximation is valid in the usual case where $T_i \ll T_{i-1}$.

This current rating I_{eff} is relevant not only for cryocoolers for microelectronic applications, but also for high-current devices such as superconducting magnets. Note that depending on the specific type of cooler, the weakest stage may not always be the coldest. This also suggests that in optimizing the design of the various stages, there may be little advantage in having substantial excess capacity (i.e., $I_i \gg I_{eff}$ for any i). On the other hand, excess cooling capacity on the first (warmest) stage might be used for other purposes, such as shielding of room-temperature thermal radiation or mounting of low-noise semiconductor amplifiers. Note also that the use of superconducting current leads (which do not obey the Wiedemann-Franz law) can in some cases enable a cryocooler to support a current larger than its rating I_{eff}.

The relative efficiency of the cooler, that is, the Power-Lead Figure of Merit, is defined by:

$$F_{PL} = I_{eff} \Big/ P_{RT} = \min\left\{ Q_i e \Big/ \left(3.6\, k_B T_{i-1}\right) P_{RT} \right\} \qquad (3)$$

or, for a more convenient dimensionless expression of the efficiency, we can use :

$$\eta_{PL} = Q_{PL}/P_{RT} = (T_0/P_{RT}) \min(Q_i/T_{i-1}) \qquad (4)$$

In some cryogenic systems with large numbers of input/output lines carrying weak signals, such as those for imaging arrays or network switches, the heat load may be dominated by heat carried on the signal leads, rather than by power leads. In such a configuration, the Joule heating is negligible whereas signal attenuation up to room temperature must be minimized. The attenuation being a function of the electrical conductivity, it is then possible to determine a minimum electrical resistance, corresponding to minimum attenuation, for each stage of the cooler, provided we know the cooling power available [4]. Since the minimum thermal

resistance between two consecutive stages can be given as $\frac{T_{i-1}^2 - T_i^2}{2Q_i}$, the minimum effective electrical resistance can be given as $L.\frac{T_{i-1}^2 - T_i^2}{2Q_i}$, where L is the usual Lorenz constant in the Wiedemann Franz law. The sum of the resistances between each stage is thus the minimum resistance for the system, and can be used as a basis for rating of the cooler. The Signal-Lead FoM can then be written:

$$F_{SL} = \left[\sum_i L.\frac{T_{i-1}^2 - T_i^2}{2Q_i}\right]^{-1} . \frac{1}{P_{RT}} \approx \left[\sum_i L.\frac{T_{i-1}^2}{2Q_i}\right]^{-1} . \frac{1}{P_{RT}} \tag{5}$$

We can note that in Eq.(5), $\left[\sum_i L.\frac{T_{i-1}^2 - T_i^2}{2Q_i}\right]^{-1}$ represents the maximum electrical conductance from room temperature to low temperature, expressed in Siemens, that the cryocooler support; a larger conductance signifies a more powerful cryocooler that can support a greater number of lines with less attenuation.

Similarly to the Power lead FoM, a dimensionless expression can be used:

$$\eta_{SL} = Q_{SL}/P_{RT} = \left[\sum_i \frac{T_{i-1}^2}{Q_i}\right]^{-1} . \frac{T_0^2}{P_{RT}} \tag{6}$$

In comparing the FoMs in Eqs. (4) and (6), both are based on weighted combinations of the heat lifts Q_i on the various stages, where the weighting factor depends on T_{i-1}, the temperature of the next warmer stage. However, for PL-FoM, the weighting goes as $1/T_{i-1}$, while for the SL-FoM, it goes as $1/T_{i-1}^2$. For a set of cryocooler stages that are ideally matched for a given application, one would then have $Q_i = K_{PL}T_{i-1}$ for the PL-FoM, with K_{PL} a fixed proportionality constant for a given cryocooler. Similarly for the SL-FOM, one would have $Q_i = K_{SL}T_{i-1}^2$, where K_{SL} is a different constant. Both of these scalings are in general accord with the tendency of cryocooler stages to be much more efficient at higher temperatures, but they give two different quantitative dependences. In designing a system for an application that has significant heat loads from both power and signal leads, an ideal cryocooler would have heat lifts that scale between the first and second power of the stage temperature.

APPLICATION TO SPECIFIC SYSTEMS AND COOLERS

Specification of a Cooler Based on the User Application

These two figures of merit can easily be used to compare cryocoolers, even if they each have a different number of stages with different intermediate temperatures. Traditionally, the choice of a cooler is done after the evaluation the potential heat load on every stage. To do so, one has to go through many approximations, such as the length and material of the power and signal lines. The calculation has to be done for each potential cooler, results are difficult to compare. Since the FoMs and their associated parameters can be easily calculated using the cryocooler manufacturers' datasheet, they are a better tool for selecting a cryocooler before purchase. In order to use these ratings efficiently, it is necessary to determine, from the application, the total bias current brought to low temperature, the maximum attenuation allowed for the signal lines, the number and design of signal lines, and the power dissipated on each stage.

As an example, we will consider a superconducting digital receiver circuit, such as the one described in [6]. The characteristics of this system are detailed in Table 1, where β is the ratio between the parallel resistance and the series resistance for a given type of transmission line.

Table 1. Characteristics of the superconducting digital receiver.

Power dissipated at 40K–70K (intermediate stage)	None
Power dissipated at 4.2 K	5 mW
DC current from room temperature to 4.2K	1.1 A
Number of signal lines needed	5 coaxes (18 GHz), 42 strip lines (1GHz)
Maximum attenuation (18 GHz)	10 dB
Maximum attenuation (1 GHz)	3 dB
β for the selected coaxial cable†	8.46
β for the selected strip line †	4.5

The only parameter not inherent to the application itself we need to fix before evaluating the FoMs is the type of signal line that will be used to carry the signals. In this example, UT47 SS (stainless steel) coaxial cable is used as well as custom-made strip lines. We can also note that this example, due to the fact that β takes into account the conductance and not the cross section only, is valid for transmission lines composed of different materials for the central conductor and the shielding, or including plating on the central conductor as it is often the case. The skin effect is important for high frequencies, as discussed in previous work [4], but the details of the calculation have not been shown here.

From Table 1, and using the FoM properties, it is easy to evaluate the requirements of the cooler needed to support this application. If we consider the PL-FoM, we can see using Eq. (2) that the FoM derives directly from the current capacity of the cooler. This current is specified by the application; the input power of the cooler is the only remaining parameter left needed to allow one to compare two systems with an equivalent current rating.

However, using the SL-FoM is a little less trivial. From Table 1, we can find the global electrical conductivity of each individual signal line by using Eq. (7)

$$S_{line} = \frac{\beta}{\alpha 2 Z_0} \tag{7}$$

Where α is the total attenuation of a high-frequency signal on the transmission line with characteristic impedance Z_0 (typically ~50 Ω).

For a total number of n lines ($n = 5$ plus $n = 42$ in Table 1) we obtain for the total conductivity requirement for the cooler

$$S_{total} = \sum_j n_j \cdot \frac{\beta_j}{\alpha_j 2 Z_0} \tag{8*}$$

where the sum on the left is over each parallel type of transmission line.

The conductivity S_{total} is directly linked to the SL-FoM (Eq.5), and the comparison of efficien-

† The attenuation, given by $\alpha = R/2Z_0$, takes into account the series resistance of a transmission line (central conductor + shielding) whereas the SL-FoM, linked to the thermal resistance by the Wiedemann-Franz law, considers the "parallel" resistance of the two conductors [4]. Strictly speaking, β represents the ratio of the ac series resistance at the signal frequency to the parallel resistance at dc. In the low-frequency limit where the conductor thicknesses are less than the ac skin depth, then one can use dc resistances for this ratio.

Then we have $\beta = (R_{cond} + R_{shield})^2/(R_{cond}R_{shield})$, and if one further assumes a uniform transmission line made from a single material, the resistivity cancels out and one can express β in terms of ratios of conductor cross sections. In the high-frequency limit where the skin depth is much less that the conductor thickness, the ac resistance would be enhanced by the ratio of the conductor thickness to the skin depth, and β would be similarly multiplied by the same factor.

* Eq. (8) here corrects an error in Equation 18 in [4].

Table 2. Requirements for a cooler supporting the superconducting digital receiver

Current rating (I_{eff})	1.1 A
Signal / conductivity rating (S_{total})	624 S
Extra cooling power at intermediate temperature	None
Extra cooling power at 4.2 K	5 mW + margin

Table 3. Specifications of commercially available coolers

Cooler model		Current rating	Signal / conductivity rating	PL-FoM	SL-FoM
First stage specification	Second stage specification				
Sumitomo SRDK-101D		5.38 A	1536 S	4.14 A/kW	1.18 S/W
5 W at 60K	0.1 W at 4.2 K				
Sumitomo SRDK-305D		32.38 A	9746 S	6.75 A/kW	2.03 S/W
20 W at 40K	0.4 W at 4.2K				
Cryomech PT-405		24.82 A	6900 S	4.6 A/kW	1.28 S/W
25 W at 65K	0.5 W at 4.2K				

The conductivity S_{total} is directly linked to the SL-FoM (Eq.5), and the comparison of efficiencies for coolers having an equivalent conductivity rating is again straight forward if we consider its input power.

The requirements for a cooler capable of handling the system specified in Table 1 are then shown in Table 2.

When evaluating a cooler, the extra cooling power needed to accommodate the heat dissipation at the lowest temperature stage itself or on intermediate stages is taken into account by subtracting this cooling power from the specified power when evaluating the cooler from its datasheet. A thorough analysis can also be done by looking at the amount of cooling power required for the SL-FoM and the PL-FoM, respectively. Note also that the required signal/conductivity rating in Table 2 is substantially increased by the skin effect. In the value presented in Table 2, the RF skin depth has been taken into account according to the method described in [4]. It is in fact approximately increased by the ratio of the conductor thickness to the skin depth.

Table 3 shows the same parameters for commercially available coolers. The figures in this table have been calculated using the datasheets provided by the manufacturers [7, 8].

Looking at Table 3, we can see that for all the coolers the performances are considerably better than the needs summarized in Table 2. Although it is not the most efficient (the FoMs are smaller than for the bigger coolers), the choice of the smallest cooler thus makes the most sense.

Optimal Choice of the Working Point of a Cooler

In the previous analysis, the FoMs and their associated properties were used to specify a cooler for a dedicated system. In [4] we showed how the FoMs can be used to compare several cryocoolers, even though they do not have the same number of stages, or are based on different technologies. In all these cases, the implicit assumption is that the data provided by the manufacturer on the datasheet are optimized for each cooler. This is not necessarily the case, and the FoMs can be used to choose the working point of an existing cooler, in order to optimize its efficiency in cooling power leads or signal leads. In this section we will first discuss the simple case of a two-stage cooler, and then extend our analysis to the 4-stage pulse tube cooler described in [9,10].

Table 4. FoM calculation details

	PL-FoM			SL-FoM		
	I_1	I_{cold}	I_{eff}	S_1	S_{cold}	S_{tot}
SRDK-101D	54.77 A	5.38 A	5.38 A	*4717 S*	*2278 S*	*1536 S*

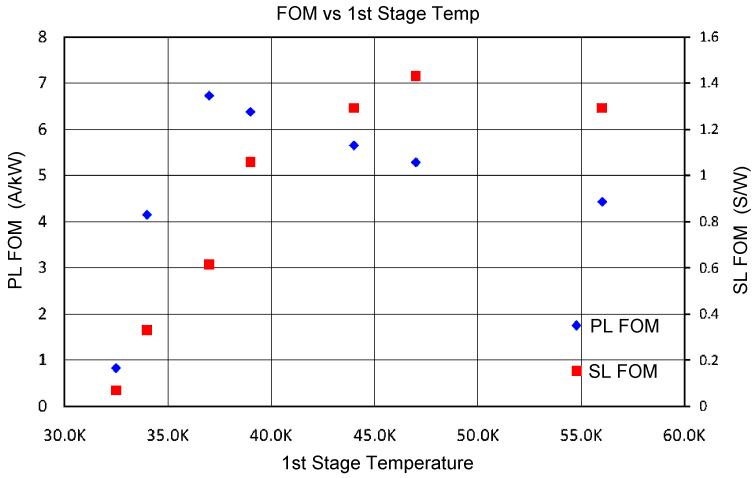

Figure 1. Optimization of the operating point

For our first case, let us consider a Sumitomo SRDK-101D [7]. According to the manufacturer's datasheet, the cooler provides 5W at 60 K, 0.1W at 4.2 K with 1.3 kW of input power. The corresponding FoMs are F_{PL} =4.14 A/W and F_{SL} = 1.18 S/W. Table 4 shows the relevant parameters for each stage of the cooler and for the system as a whole.

It is interesting to note that in both cases, the individual performances of the first and second stage are very different. For the PL-FoM the extra current capacity clearly shows that the system is unbalanced, and that its PL-FoM could be improved by lowering the temperature of the first stage, thus with a lower cooling power. Knowing the load map of the cooler, we can plot the PL-FoM versus the temperature of the first stage. A similar plot can be done for the SL-FoM. These plots are shown in Figure 1.

As we can see in Figure 1, there is a clear influence of the operating temperature of the cooler on the FoM. It is also notable that optimum operating points for the SL-FoM and the PL-FoM are not identical. This can also been seen from the expressions of the FoMs themselves, (Eqs. 2 and 4), which indicate that the cooling power on the ith stage should scale as T_{i-1} for the PL-FoM and T_{i-1}^2 for the SL-FoM.

For the 4-stage cooler developed by Lockheed Martin for Hypres [9], the analysis is more complicated. As we showed in a previous study [10], the cooling powers available on each stage are interconnected. An increase of the temperature on the second stage for example will result in a decrease of the available cooling power on the third stage. The phenomenon tends to indicate that one cannot optimize the temperature of each stage individually. As free adjustments of all the parameters would lead to a very complicated optimization process, we chose to set the operating temperature of the first stage at 66 K. We made this practical choice because we observed that any higher temperature will tend to degrade significantly the performances of the other stages whereas at lower temperatures we observed some instabilities in the system. The remaining parameters are then the temperatures of the second and third stages.

It is thus possible to plot the different FoMs versus these temperatures. Those graphs are presented in Figures 2 and 3.

As for a two-stage cooler, there appears to be a strong dependence of the PL-FoM and SL-FoM on the temperature of the intermediate stages. The optimum is also different for the

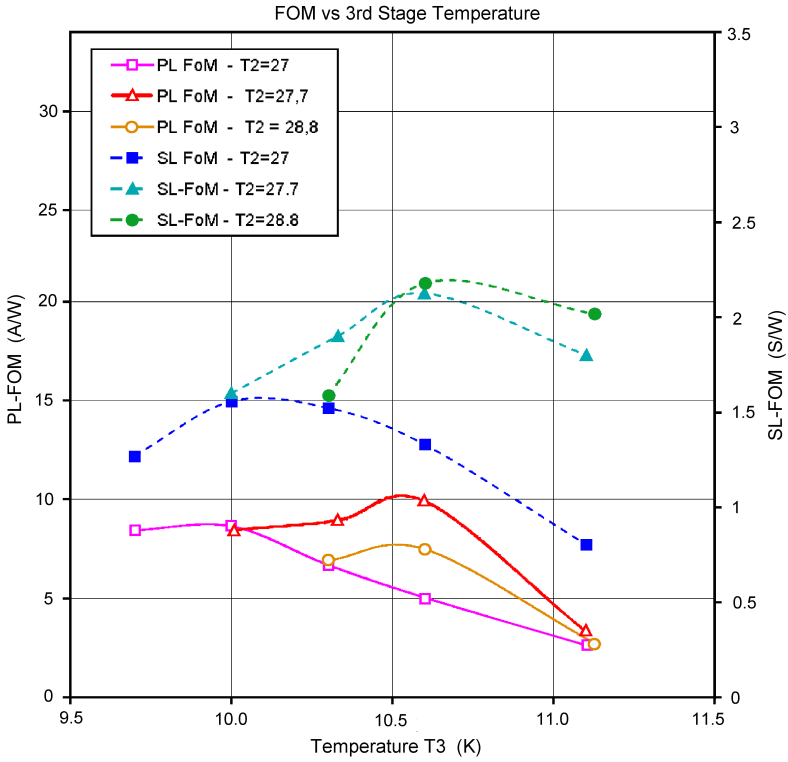

Figure 2. FoM third stage temperature dependence

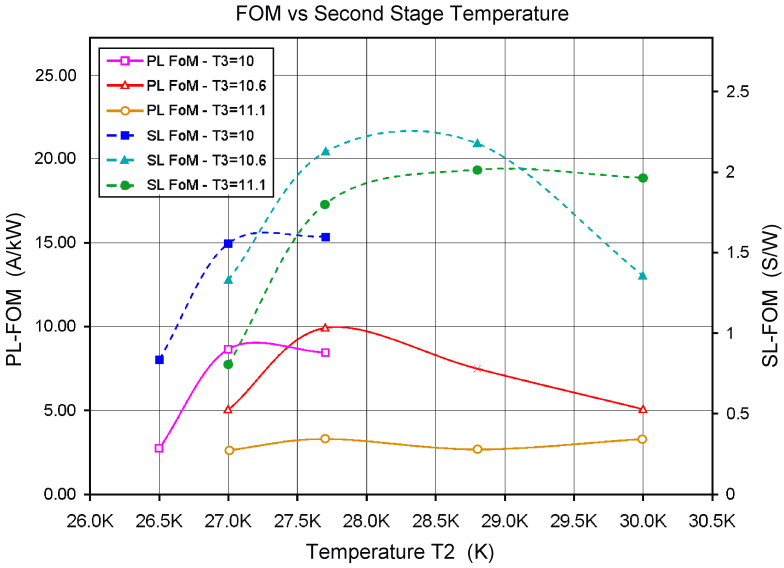

Figure 3. FoM second stage temperature dependence

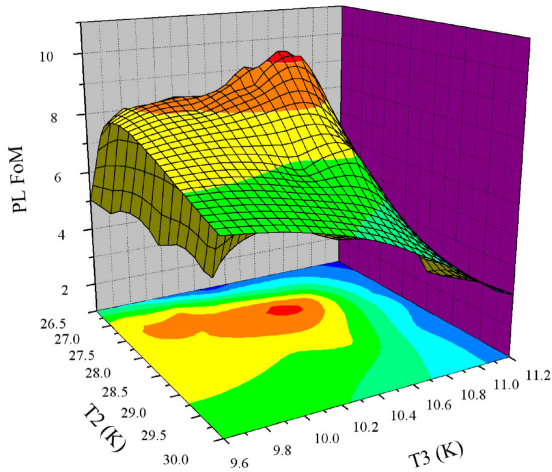

Figure 4. PL FoM as a function of both third stage and second stage temperatures

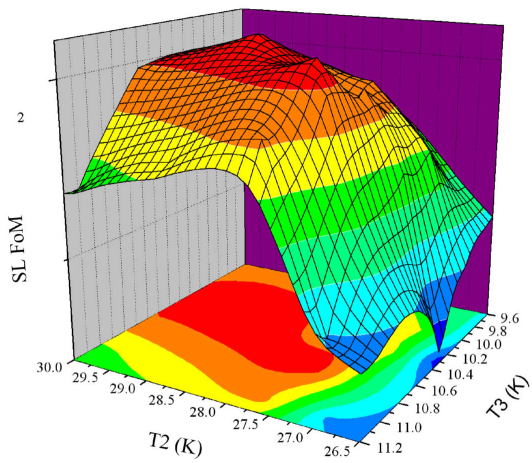

Figure 5. SL FoM as a function of both third stage and second stage temperatures

SL-FoM and the PL-FoM. The overall optimum operating temperature cannot be determined from these graphs, because of the interdependence of the stages. In order to have a clear map of the performances of the cooler, we interpolated the behavior of the stages of the cooler. As a result, a three-dimensional plot of the SL-FoM and PL-FoM could be made. Those plots are presented in Figures 4 and 5.

A maximum is now clearly visible, both for the PL-FoM and SL-FoM. It is interesting to note that the overall shape of the temperature dependence of the two FoMs is different; this is of course explained by the analytical expression of the FoMs, showing T and T^2 temperature dependence for PL-FoM and SL-FoM respectively.

CONCLUSION

We have proposed two new figures of merit allowing a fair comparison of the thermal performance of different types of multi-stage cryocoolers, for cryogenic applications dominated by heat flow on power leads or on signal leads. We have shown that these figures of merit and their associated properties can be used to match a cryocooler to a predefined application, as well as to determine the optimum operating point for a cooler. This analysis should

prove helpful in assessing the comparative performance of cooling stages both within a given cooler, and between different coolers, and may be used as guidelines in the design of new cryocoolers and cryopackages for a variety of applications of superconducting and cryogenic devices.

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