Real and Reactive Flows in Regenerative Cryocoolers

Vincent Kotsubo

Department of Physics
University of Colorado, Boulder, CO 80309

ABSTRACT

The flows in a regenerative cooler can be separated into two orthogonal components, one in-phase and the other in-quadrature with the pressure amplitude. The in-quadrature component generates the pressure amplitude, essentially compression of gas in a can, while the in-phase component time-averages with the pressure amplitude to produce acoustic power. This separates the compliant, or gas spring behavior, from the power conversion behavior of the cooler, and provides insight into some of the processes within the cooler. Taking the example of an inertance tube pulse tube with a resonant piston compressor, the pressure amplitude is seen to be generated through resonance with the piston mass and the effective mass of the inertance tube, which avoids loss of energy of compression each cycle. The in-phase component is deamplified by the regenerator in proportion to the ratio of the cold and warm end temperatures, which is a direct consequence of conservation of mass and the ideal gas law. This leads to a corresponding decrease in acoustic power, which then leads to Carnot's COP. A key point of this paper is the recognition that the pressure amplitude and the real flow are the fundamental thermodynamic quantities, while the reactive flow serves to generate the pressure amplitude and is not fundamental to the cooling process. Therefore, the pulse tube designer has considerable flexibility in methods used to generate and manage reactive flows.

INTRODUCTION

Regenerative coolers are driven by oscillating flows and pressures, which generate acoustic power, which in turn, drive thermodynamic processes that produce cooling. These processes are well understood and have been thoroughly discussed in the literature[1,2,3].

However, processes specific to the flows and pressures themselves have not been illuminated in as much depth, so this paper presents a methodology to examine these underlying processes, and presents conclusions arising from this methodology. The approach is to separate the flows into real and reactive components, i.e. components that are in-phase and in-quadrature, respectively, with the pressure amplitude. This then leads to the observation that coolers consist of gas-spring/mass systems which generate the pressure amplitude, plus a real flow component that contributes directly to the acoustic power. The gas spring behavior is of particular significance, as the management of energy stored in gas compression impacts cooler efficiency.

This presentation is based on lumped-element thermoacoustics[1], where two of the main components the regenerator and thermal buffer tube (TBT)[4], are simply compressible volumes. As such, relations between pressure amplitudes and volume flow rates are simple, and allow deconstruction of the flows into real and reactive parts.

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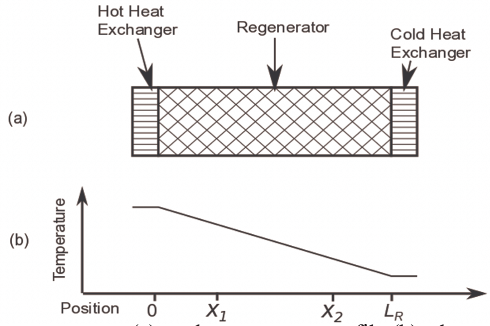

Figure 1. Schematic of the regenerator (a), and temperature profile (b), along with relevant coordinates.

For specificity, this paper will focus on a pulse tube cryocooler driven by a resonant piston compressor[5] and terminated by an inertance tube[1]. This configuration is common in small, compact coolers that operate at frequencies in the 30 to 60 Hz range. Other types of coolers such as Stirlings, Gifford-McMahons, and valved pulse tubes do not use inertance tubes, and the latter two coolers do not use resonant pistons. However, many of the concepts presented are easily adapted to these other systems.

The standard idealizations are used, complete local isothermalization in the regenerator, complete adiabaticity in the TBT, no viscous loss except in an orifice, and zero longitudinal thermal conductivity.

PRESSURE AND FLOW

Pressures and flows are both taken to be sinusoidally time varying quantities. Pressure consists of the average pressure P_0, plus an oscillating part, and is given by:

$$P(t) = P_0 + P_1 \cos(\omega t) \tag{1}$$

where P_1 is the oscillatory pressure amplitude and ω is the angular frequency. It is independent of position, in part because the wavelength of sound is much longer than typical cooler dimensions, and in part because of the idealization of zero viscous pressure drop.

Flow is described by the volume flow rate, $U(x,t)$, which is:

$$U(x,t) = U_1(x)\cos[\omega t + \phi(x)] \tag{2}$$

where U_1 is the volume flow rate amplitude given by:

$$U_1(x) = A_C v(x) \tag{3}$$

where A_C is the cross-sectional area of the particular component and $v(x)$ is the local fluid velocity. Equation 2 can be thought of as the swept volume rate of an imaginary piston at position x. The phase of the volume flow rate relative to P_1 is $\phi(x)$, and P_1 is chosen to be real. Both the amplitude and phase of $U(x,t)$ are functions of position.

REACTIVE FLOWS, GAS SPRING

Regenerator

This analysis begins by deriving the relationship between the volume flow rate and pressure amplitude in the regenerator. The regenerator, as shown in Figure 1, has a length L_R, cross-sectional area A_C, and temperature profile $T(x)$. The total mass between any two points x_1 and x_2 can be calculated by simply integrating the ideal gas law between those two points:

$$m = \frac{M}{R} P(t) \int_{x_1}^{x_2} A_C \frac{dx}{T(x)} \tag{4}$$

where M is the molar mass and R is the gas constant. Taking the time derivative of Equation 4:

$$\frac{U_1(x_1)}{T_0(x_1)} \cos[\omega t + \phi(x_1)] - \frac{U_1(x_2)}{T_0(x_2)} \cos[\omega t + \phi(x_2)] = \frac{-\omega P_1 \sin(\omega t)}{P_0} \int_{x_1}^{x_2} A_C \frac{dx}{T(x)} \tag{5}$$

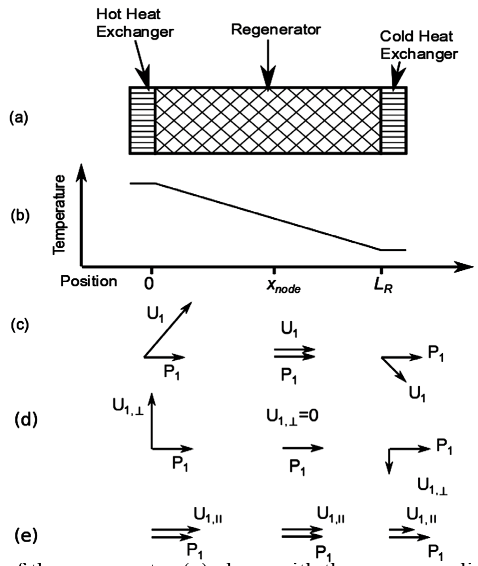

Figure 2. Schematic of the regenerator (a) along with the corresponding temperature profile (b). The volume flow phasors are shown for the warm end, cold end, and node, with the total phasor shown in (c), the reactive components that generate P_1 in (d), and the real components that generate acoustic power in (e).

where Equation 1 for $P(t)$ has been used and, the mass flow rate at any point is converted to volume flow rate using the relation:

$$\frac{dm}{dt} = \rho_0 U_1 = \frac{P_0 M}{R T_0} U_1 \tag{6}$$

where ρ_0 is the local average density and T_0 is the local temperature. Equation (5) is actually two orthogonal equations, one with volume flow rates in-quadrature with P_1 (Equation 7) and the other with volume flow rates in-phase with P_1 (Equation 8):

$$\left[\frac{U_{1,\perp}(x_1)}{T_0(x_1)} + \frac{U_{1,\perp}(x_2)}{T_0(x_2)}\right] sin(\omega t) = \frac{-\omega P_1}{P_0}\left[\int_{x_1}^{x_2} A_C \frac{dx}{T(x)}\right] sin(\omega t) \tag{7}$$

and

$$\left[\frac{U_{1,\|}(x_1)}{T_0(x_1)} - \frac{U_{1,\|}(x_2)}{T_0(x_2)}\right] cos(\omega t) = 0 \tag{8}$$

The notation has been simplified by using the subscripts $\perp$ and $\|$ to indicate components in-quadrature (reactive) and in-phase (real) with P_1:

$$U_{1,\perp}(x) \equiv U_1(x)\, sin[\phi(x)] \tag{9}$$

and

$$U_{1,\|}(x) \equiv U_1(x)\, cos[\phi(x)] \tag{10}$$

The $sin(\omega t)$ and $cos(\omega t)$ functions are, of course, inconsequential in Equations 7 and 8 but are left in as a reminder of the process used to obtain these equations. Also, these equations relate the amplitudes of P and U. The oscillating quantities, including phase information, are still expressed by Equations 1 and 2.

Equation 7 indicate that P_1 is generated only by the reactive components of U_1, which are 90 degrees out of phase with P_1. This is simply the behavior of compressing gas in a can with a piston. Equation 8 show that real components of U_1 cannot generate P_1 since the real components do not allow local mass buildup. Mass simply "sloshes" back and forth in the real components.

Phasor Description

The commonly used phasor diagram can be used to illustrate Equations 1 and 2 with amplitudes determined by Equations 7 and 8. Most well optimized coolers have a regenerator phasor diagram[6] similar to that shown in Figure 2, where at the warm end, U_1 leads P_1 by very roughly 45 degrees,

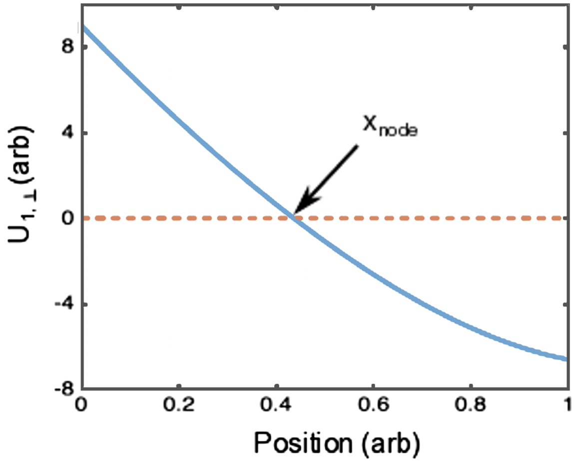

Figure 3. Example of reactive flows as a function of position assuming a linear temperature gradient and flows at the warm and cold ends 180 degrees out of phase. The reactive flow is zero at some point within the regenerator, and the location depends on the relative reactive flow amplitudes and regenerator temperatures at the ends.

and at the cold end, lags P_1 by roughly 45 degrees (Figure 2c). Equation 7 represents the components of U_1 perpendicular to P_1, as shown in Figure 2d, while Equation 8 represents the components of U_1 parallel to P_1, as shown in Figure 2e.

The Dual Gas Spring

Given the temperature gradient and reactive flows at any two points within the regenerator, the reactive flows everywhere can be calculated using Equation 7. An example is shown in Figure 3, where a linear temperature gradient has been assumed. If the reactive flows are 180 degrees out of phase at the ends of the regenerator, then, at some point in the regenerator, the reactive flow must be zero. This point is referred to as the "node". In this paper, the side of the cooler on the compressor side of the node is the "upstream" side while cold side of the node is the "downstream" side.

The fact that the upstream flows are 180 degrees out of phase with the downstream flows, and that at the node, the reactive flow is zero, i.e. no reactive mass flows past this point, indicate that P_1 upstream of the node is generated by mass flow from an upstream source, while P_1 downstream of the node is generated by mass flow from a downstream source.

At the node, the zero-flow condition indicate that, regarding the reactive flows, it is essentially solid boundary. In effect, the regenerator can be separated into two independent volumes. Upstream of the node the relation between $U_{1,\perp}$ at the warm end (x = 0), and P_1 is:

$$\frac{U_{1,\perp}(0)}{T_0(0)} = \frac{\omega P_1}{P_0}\left[\int_0^{x_{node}} A_C \frac{dx}{T(x)}\right] \tag{11}$$

while downstream of the node:

$$\frac{U_{1,\perp}(L_R)}{T_0(L_R)} = \frac{-\omega P_1}{P_0}\left[\int_{x_{node}}^{L_R} A_C \frac{dx}{T(x)}\right] \tag{12}$$

This represents two independent gas springs, one facing upstream and the other facing downstream.

Thermal Buffer Tube

The relation between P_1 and U_1 in the TBT can easily be derived from the condition that compression is adiabatic, such that:

$$PV^{\gamma} = constant \tag{13}$$

where γ is the ratio of isobaric to isochoric specific heats. Taking time derivatives, the reactive terms yields:

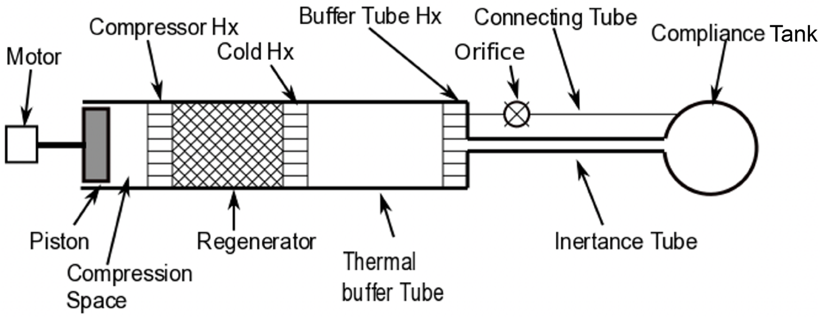

Figure 4. Schematic of a standard inertance tube pulse tube.

$$\left[U_{1,\perp}(0) - U_{1,\perp}(L_{TBT})\right] = \frac{\omega V_{TBT} P_1}{\gamma P_0} \tag{14}$$

where V_{TBT} is the volume of the thermal buffer tube, $x = 0$ is the location of the cold end of the buffer tube, and $x = L_{TBT}$ is the location of the warm end of the buffer tube. As with the regenerator, this is a gas spring.

The real flows at the ends of the TBT can also be similarly obtained from taking the time derivative of Equation 13 with $P_1 = 0$:

$$U_{1,\parallel}(0) = U_{1,\parallel}(L_{BT}) \tag{15}$$

The Gas Spring and Inertance Tube

To analyze the inertance tube and its interaction with the gas-spring, consider the standard pulse tube configuration shown in Figure 4. It consists of a motor driven, finite-mass piston, compression space, heat exchangers, regenerator, TBT, inertance tube, orifice, and compliance tank. For this analysis the heat exchanger and compression space volumes are considered negligible. At the warm end, the inertance tube and orifice terminate the TBT, and in this idealized model the inertance tube is dissipationless, all dissipation is assigned to the orifice.

The boundary condition of continuity of volume flow between the cold end of the regenerator and TBT results in a single gas spring consisting of the regenerator volume downstream of the node and the volume of the TBT. The warm end of the TBT is coupled to the orifice and inertance tube. The orifice, being dissipative, couples to the real flow, while the inertance tube couples to the gas-spring.

The inertance tube has an effective mass equal to:

$$m_{eff} = \rho_0 l_{IT} A_{IT} \tag{16}$$

where l_{IT} and A_{IT} are the length and cross-sectional area of the inertance tube. This effective mass is driven by a force produced by the oscillating pressure amplitude of the combined regenerator/TBT gas spring:

$$F = P_1 A_{IT} \tag{17}$$

Thus, the gas-spring and effective mass combination forms the standard acoustic Helmholtz resonator[7]. On resonance, the reactive response of the inertance tube is the source that drives P_1 in the TBT and in the downstream section of the regenerator.

The Gas Spring and Compressor Piston

Turning to the compressor piston, the piston mass and the gas-spring formed by the regenerator volume upstream of the node also form an oscillator. However, since U_1 of the piston consist of both real and reactive components, and only the reactive component interacts with the gas spring, the flow components must be separated in order to understand the mass/spring system. The piston

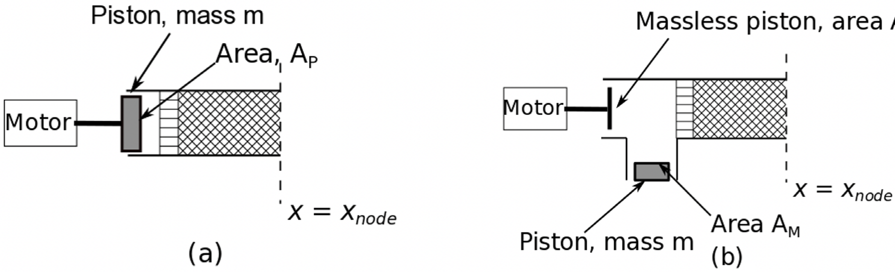

Figure 5. The compressor piston can be separated into two pistons, one massless and driven by the motor, and the other with finite mass and is coupled to the gas spring. The areas of the two pistons reflect the in-phase and in-quadrature flow from the original model.

in fact can be conceptually split into two pistons, one that is massless and driven by the motor, and the other a finite-mass piston coupled to the gas spring. This is depicted in Figure 5. It can be shown[8] that if the areas of the massless and finite mass pistons, A_F and A_M, are:

$$A_F = cos(\phi_P)\,A_P \tag{18}$$

and

$$A_M = sin(\phi_P)\,A_P \tag{19}$$

where A_P is the area of the original piston, the behavior of the entire system is replicated. On resonance, the velocity of the massless piston will be purely real, while the velocity of the finite-mass piston will be purely reactive relative to P_1. The amplitudes of each piston in the two-piston configuration will be the same as the amplitude of the original piston; the power delivered by the motor is identical in both cases, and the pneumatic response of the pulse tube is also identical. In Equations 18 and 19 ϕ_P is the phase of the single piston velocity.

To summarize, the reactive behavior of the pulse tube is that of two mass/gas-spring resonant systems, one upstream of the node, and the other downstream of the node, as shown in Figure 6. The node location depends on multiple factors, including volumes, temperatures, inertance tube dimensions, piston mass and surface area, and operating frequency.

On resonance, energy reciprocates between the spring systems and the mass, so the energy of gas compression is recovered each cycle. This is a highly efficient method of generating P_1.

In contrast, valved compressors, namely those used for Gifford-McMahon coolers or valved pulse tubes that operate at 1 Hz, simply dissipate the energy of compression each cycle. Thus, they have an inherent loss mechanism. There is a misconception that the loss is caused by pressure drop in the valves, when in fact, the pressure drop is inherent in the timing in opening the valves relative to the operating cycle of the coldhead itself.

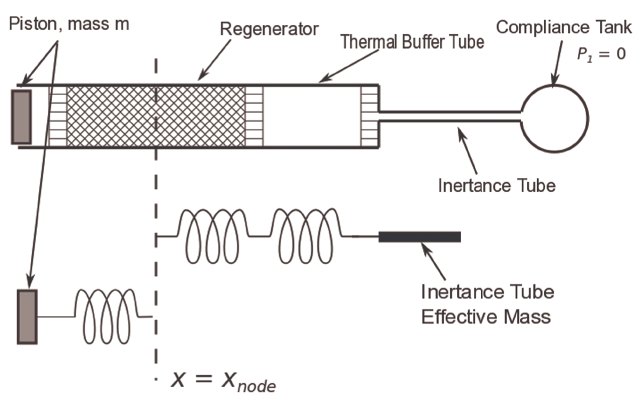

Figure 6. The reactive behavior of the pulse tube is that of two gas springs coupled to two masses, one at each end of the pulse tube. The gas springs are joined at the node, which is a hard boundary.

The previous discussion on replicating a single piston with two pistons, one reactive and one real, indicate that the mechanism that generates P_1 does not have to be the same as the mechanism that generates $U_{1,\|}$. Therefore, in principle, a resonant piston could be attached to the gas space in a valved compressor to recover energy of compression. However, the low frequency of operation would require a mass too large to be practical. Also, there is no fundamental limitations on the location of the piston, so in fact, if the piston were dissipationless, it could connect to the gas space anywhere.

Multi-Staged Coolers

As discussed above, reactive mass flow from the compressor does not pass the node in the regenerator, so in the case of a two-stage inertance pulse tube, P_1 upstream of the node of the 2nd stage regenerator is generated by resonating with the 1st stage inertance tube. The effective mass of the 1st stage inertance tube forms a resonant system with a gas spring consisting of the 1st stage TBT volume, the volume downstream of the node in the 1st stage regenerator, and the volume upstream of the node in the 2nd stage regenerator. This is significantly less lossy compared to having the 2nd stage regenerator pressurized by mass flow from the compressor, which would require flow through the entire 1st stage regenerator. This is one of the factors that result in improved COP in multi-staged pulse tubes.

REAL FLOW AND ACOUSTIC POWER IN THE REGENERATOR

The acoustic power $\dot{E}_2$ is one of the significant quantities in thermoacoustics[1], and there are a number of excellent publications discussing its application to coolers. It is defined as:

$$\dot{E}_2(x) = \langle PU(x) \rangle \tag{20}$$

where the brackets indicate time averaging. Using Equations 1 and 2 for *P* and *U(x),* evaluating the time average, and using the nomenclature of Equation 9, the expression for acoustic power simplifies to:

$$\dot{E}_2(x) = \frac{1}{2} P_1 U_{1,\|} \tag{21}$$

The acoustic power has two contributions P_1, which is arises from the reactive flow, $U_{1,\|}$ and , the real flow itself.

Since P_1 is constant everywhere, the temperature dependence of the acoustic power in the regenerator is easily obtained starting from Equation 8, which shows that $U_{1,\|}$ is proportional to temperature. At the warm and cold ends of the regenerator the volume flow rates $U_{1,\|,\mathrm{hot}}$ and $U_{1,\|,\mathrm{cold}}$ are related by:

$$\frac{U_{1,\|,hot}}{T_{0,hot}} - \frac{U_{1,\|,cold}}{T_{0,cold}} = 0 \tag{22}$$

where $T_{0,\mathrm{hot}}$ and $T_{0,\mathrm{cold}}$ are the corresponding average temperatures. The relationship between the acoustic power at the ends of the regenerator are now:

$$\frac{\dot{E}_{2,cold}}{\dot{E}_{2,hot}} = \frac{T_{0,cold}}{T_{0,hot}} \tag{23}$$

Acoustic power is deamplified by the regenerator by the ratio of the cold to hot end temperatures. This is a direct result of the deamplification of the real volume flow rate, which is a consequence of the ideal gas law and the fact that there is no mass buildup associated with the real flow within the regenerator. Numerous other publications[3] have shown that Equation 23 leads to Carnot's COP for coolers that recover the acoustic power that exits the cold end of the regenerator.

INERTANCE TUBE AND ORIFICE

In the pulse tube configuration discussed in this paper, the warm end of the TBT is terminated by a combination of a dissipative element, the orifice, and a reactive element, the inertance tube.

These two components have orthogonal functionalities. The orifice must be sized to dissipate the acoustic power from the TBT, and this power is determined by the power supplied by the compressor and the subsequent deamplification by the regenerator. The inertance tube effective mass, on the other hand, controls the location of the node, which is one of the factors that affect overall efficiency in real coolers when losses are considered. The efficiency gain from the optimum node location is entirely due to the optimum reactive flow distribution.

SUMMARY

The compressible volume nature of the regenerator and thermal buffer tube allow the volume flow rate to be decoupled into reactive and real components. The reactive component generates the pressure amplitude and is associated with the gas spring, while the real component produces acoustic power by direct multiplication with the pressure amplitude.

By examining the reactive component only, the regenerator/TBT combination is shown to be two gas springs- one upstream of the node, connected to the piston mass, and the other downstream of the node, connected to the effective mass of the inertance tube. Both of these uses resonance to generate P_1, which recaptures the energy of gas compression.

The TBT is terminated by two orthogonal components, the orifice and inertance tube. The orifice dissipates acoustic power and associated with the real flows, while the inertance tube is one of many components that determine the location of the node in the reactive flows. It is managing the reactive flow components where the COP gain is achieved using the inertance tube.

Valved compressors do not recapture the energy of gas compression, which results in a lower efficiency compression cycle. In two-stage coolers, the pressure amplitude upstream of the node in the second-stage regenerator comes from resonance with the inertance tube of the first stage, which is more efficient than having the mass flow come from the compressor through the first stage regenerator.

The regenerator deamplifies the real volume flow rate according to the temperature ratio of the cold to warm ends, which leads to Carnot's C.O.P. when considering the regenerator only and is directly a result of considering the real flow only, mass conservation, and the ideal gas law.

Equation 21 embodies one of key implications of this paper. The thermodynamics of cooling is contained in P_1 and the real flow $U_{1,\|}$. The reactive flow, $U_{1,\perp}$, is not directly involved in the thermodynamics. There is no thermodynamic restriction on generation of reactive flows, so the objective in optimizing a cooler is to configure the reactive flows such that it generates P_1 in the most efficient manner possible. The real flow, on the other hand, being the fundamental thermodynamic quantity, must originate at the compressor piston, get deamplified by the regenerator, and dissipate all power at the orifice.

ACKNOWLEDGMENTS

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